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On trapped oscillations in a slightly viscous rotating fluid

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Some properties of trapped oscillations in an inviscid rotating fluid were studied by Stewartson (1971, 1972). In this paper the modifications necessary to allow for a small viscosity in the fluid are discussed. It is found that concentrated disturbances of the kind used in ray theory are only possible if the rays are thought of as emanating from two sources outside the fluid and at opposite ends of the ray. Otherwise the ray must have a lateral tail, which vitiates the notion of a ray. The double-source character of rays is preserved at reflexions except when at glancing incidence. In this case a substantial part of the ray strength is lost and may give rise to a lateral tail if uncompensated. The focusing of the rays at the limit cycle is also discussed and shown to lead to a catastropic loss of energy, so that a free oscillation if set up rapidly decays. A simple forced oscillation is considered and shown to lead to weak resonances in the bands of trapped oscillations.

1. Introduction

The oscillatory motions superposed on a fluid rotating in a cavity as if solid have received much attention in recent years. A recent review of the work has been given by Greenspan (1968) and is principally concerned with interior problems, i.e. when the fluid is bounded externally by a rigid closed surface. The principal effect of a weak viscosity is then confined to the Ekman layer near the outer boundary of the fluid and its relative contribution to the fluid motion is of $O(E^{\frac{1}{2}})$, where $E = \nu/\Omega a^2$, ν being the kinematic viscosity, Ω the angular velocity and $2a$ a representative diameter of the cavity. Over a narrow region near the critical circles these layers thicken slightly but overall it appears at first sight that, in many problems of geophysical and oceanographic interest, viscous effects are not important.

A start has also been made on the study of oscillatory motions when the fluid has an internal boundary. Such studies are relevant to geophysics; thus the oscillations of a fluid confined in a spherical shell are of interest in connexion with motions in the earth's core and the geomagnetic secular variation (Hide & Stewartson 1972), now that the existence of the central body has been firmly established. The analytic studies have not yet made much progress beyond consideration of such oscillations when the shell is thin, but it has already become evident that the presence of the inner boundary can exert an important effect on these oscillations. In particular the characteristic cone which touches the inner sphere (at the critical circle) is seen to be linked with a pathological element in the inviscid motions. This was first noticed as a relatively weak phenomenon in asymmetric oscillations (Stewartson & Rickard 1969) but has subsequently been found to be a more pronounced feature of symmetric trapped oscillations. The existence of such oscillations was first suggested by Stern (1963) and closed ray-patterns were found shortly afterwards by Bretherton (1964). More recently, Israeli (1971) has pointed out that there are closed ray-patterns for a continuous range of frequencies near that which leads to Bretherton's closed ray-pattern. Stewartson (1971, 1972, subsequently denoted by Sa, b) demonstrated that, apart from one exceptional case, all other rays in the trapped zone converge onto these limit cycles and that pathological features then develop in the flow field.

Israeli has also made a numerical study of symmetrical oscillatory motions in spherical shells taking into account the effects of viscosity, and with particular reference to trapped oscillations. He finds good agreement with the theoretical value of the period necessary for one of the Bretherton limit cycles. The period of the trapped oscillation is identified by noting the peak value of some representative quantity as the period of the forcing disturbance is varied; this peak becomes less pronounced as the thickness of the shell diminishes. The details of the flow properties at the peak are also only vaguely similar to the analytical predictions for an inviscid fluid, although this may be due to the nature of the forcing oscillation (a sinusoidal variation in the angular velocity of one of the spheres), which must generate fluid motions over the whole of the shell. Israeli also makes a favourable comparison with the experimental values of the resonant frequencies obtained by Aldridge (1967) for a shell whose inner radius was half that of the outer.

In this paper we make a start on the theory of oscillatory motions of a viscous fluid, restricting attention to a thin shell and a very weak viscosity. The inviscid studies make considerable use of ray theory and we find that, if this theory is to be applicable in a weakly viscous fluid, each ray must be thought of as being generated by two equal sources, placed on the lineal extensions of the ray beyond the fluid, one on each side. The implication is that a disturbance cannot be thought of as propagating along the ray in a particular direction, from the inner sphere to the outer sphere for example, but must be thought of as being composed of two disturbances, essentially equal in strength, propagating in opposite directions. Otherwise the ray has an extensive lateral tail which contradicts the notion of a ray. At each reflexion of the ray, either at the inner or the outer sphere, the origins of the two equivalent sources move, remaining outside the fluid of course, but as the ray approaches the limit cycle they converge towards the two points where the meridional sections of the characteristic surfaces touch the inner sphere. Hence, as expected, the oscillations must decay if there are no sources in the fluid or on the boundaries, and indeed a rough argument shows that the rate of decay is on the inviscid time scale. An examination of the resonant wave amplitudes produced by a particular forcing motion of the boundaries strongly suggests that the maximum amplitudes in the majority of the trapped wave do tend to infinity as $\nu \to 0$ but only very slowly.

2. Governing equations

We consider a viscous incompressible fluid of density ρ and kinematic viscosity ν confined between two concentric spheres of radii a and b ($a > b$). In the basic motion the spheres and the fluid rotate as if solid about a common diameter $\mathbf{l}$ with angular velocity $\mathbf{\Omega}$. The motion is slightly disturbed, the velocity perturbations $\mathbf{u}$ being small so that the governing equations may be linearized and written in the form

$$\operatorname{div}\mathbf{u} = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + 2\mathbf{\Omega}\times\mathbf{u} = -\frac{1}{\rho}\operatorname{grad}p - \nu\operatorname{curl}\operatorname{curl}\mathbf{u}, \tag{2.1}$$

where p is the pressure. Let us define a reduced set of co-ordinates, in terms of the usual spherical co-ordinates (r, θ, ϕ), based on the common centre of the spheres and the axis $\mathbf{l}$, by writing

$$r = b(1+\epsilon z), \quad \theta = \tfrac{1}{2}\pi - \epsilon^{\frac{1}{2}}y, \quad \epsilon = (a-b)/b, \tag{2.2}$$

so that the inner spherical boundary is defined by $z = 0$ and the outer by $z = 1$. Further, the equatorial plane, perpendicular to $\mathbf{l}$, is given by $y = 0$.

Now we suppose that $\epsilon \ll 1$, so that the fluid is confined with a thin shell, take the motion to be axisymmetrical, and therefore independent of ϕ, and to be sinusoidal in time. Specifically we write

$$\mathbf{u} = \Delta a\Omega(\epsilon^{\frac{1}{2}}u_r, u_\theta, u_\phi)\exp 2i\Omega\,\epsilon^{\frac{1}{2}}\omega t, \tag{2.3}$$

where u_r, u_θ and u_ϕ are functions of y and z only and it is understood that the real parts of the complex functions are to be taken. Thus we are interested in relative fluid motions confined to the immediate vicinity of the equator. The reduced frequency ω is to be found and Δ is a small constant, satisfying the condition $|\Delta| \ll \epsilon$ as a necessary condition for the validity of the linearization of the equations. We shall see in § 5 below that this condition is not sufficient. Setting

$$p = 2\rho\Omega b\epsilon P(y,z) + \text{constant}, \tag{2.4}$$

substituting into (2.1) and neglecting all but the lowest powers of ϵ, we find that

$$\partial u_r/\partial z = \partial u_\theta/\partial y, \quad u_\phi = \partial P/\partial z, \tag{2.5a,b}$$

$$2(i\omega u_\theta - y u_\phi) = 2\,\partial P/\partial y + R^{-1}\,\partial^2 u_\theta/\partial z^2, \tag{2.5c}$$

$$2(i\omega u_\phi + y u_\theta) = R^{-1}\,\partial^2 u_\phi/\partial z^2, \tag{2.5d}$$

where

$$R = (\Omega b^2/\nu)\,\epsilon^{\frac{5}{2}}. \tag{2.6}$$

These equations may be simplified on writing

$$u_\theta + i u_\phi = \partial Z_+/\partial z, \quad u_\theta - i u_\phi = \partial Z_-/\partial z, \tag{2.7}$$

$$2u_r = \partial(Z_+ + Z_-)/\partial y, \tag{2.8}$$

when they reduce to

$$\frac{1}{R}\frac{\partial^3 Z_\pm}{\partial z^3} = 2i(\omega \pm y)\frac{\partial Z_\pm}{\partial z} \pm 2i\frac{\partial Z_\pm}{\partial y}. \tag{2.9}$$

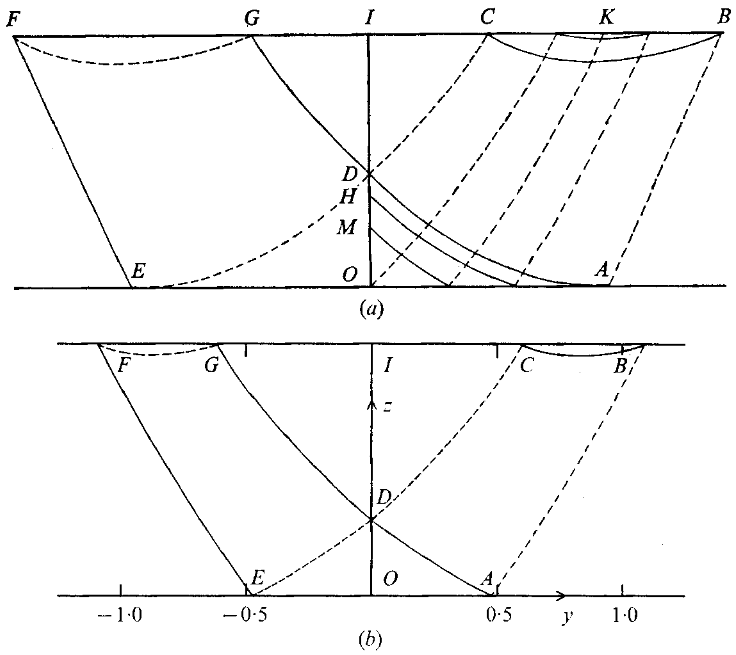

FIGURE 1. Closed ray-patterns ($ABCDEFGDA$) for the trapped mode corresponding to (a) $\omega = \omega_1$ $(= \frac{2}{3}\sqrt{2})$ and (b) $\omega = 0{\cdot}85$.

These equations were first written down by Roberts & Stewartson (1963) in another but related situation and have subsequently been re-derived by a number of authors. The equations are valid provided that (i) the solution obtained is confined within a finite distance of the line $y = 0$, i.e.

$$Z_{\pm} = 0 \quad \text{if} \quad |y| > y_0 \tag{2.10}$$

for some positive y_0, or at least $|Z_{\pm}| \to 0$ sufficiently rapidly as $|y| \to \infty$, and (ii) $R \geqslant 1$. The first condition implies that the relative motion of the fluid is confined within a small angular distance of the equator and so the waves are trapped, while the second is the mathematical statement of an assumption of small viscosity.

For $R = \infty$ the possible existence of such trapped waves has been discussed by Bretherton (1964) and some of their properties elucidated in Sa and Sb. The governing equations (2.9) then have a solution

$$Z_+ = f(\tfrac{1}{2}(\omega+y)^2 - z), \quad Z_- = g(\tfrac{1}{2}(\omega-y)^2 - z), \tag{2.11}$$

where f and g are arbitrary. For free oscillations $u_r = 0$ on $z = 0$, 1 and hence, from (2.8), $f+g = 0$ on $z = 0$, 1. Using ray theory it is now possible to determine the values of ω which permit $Z_{\pm}$ to be non-zero in a finite region only and to find some of their properties. It emerges that the values of ω are not discrete but lie in bands of which the lowest is $(\frac{2}{3})^{\frac{1}{2}} \leqslant \omega \leqslant \frac{2}{3}\sqrt{2}$ and in general $\Omega_n \leqslant \omega \leqslant \omega_n$, where $\omega_n^2 \approx \frac{3}{2}n - 0{\cdot}81$ when n is a large integer while $\omega_n - \Omega_n = O(\omega_n^{-3})$ when ω_n is large. At each permissible value of ω it is possible to find a closed ray-pattern; also any neighbouring ray inside it will, if it is followed and allowance is made for reflexions at the boundaries, converge on this closed pattern, which thus plays the role of a limit cycle. The upper limit $\omega = \omega_n$ corresponds to a closed ray-pattern which touches the inner sphere ($z = 0$) at the points $(\pm\omega, 0)$. The lower limit $\omega = \Omega_n$ corresponds to a closed ray-pattern which meets the outer sphere ($z = 1$) at the points $(\pm\omega, 1)$. The lowest permissible value of ω_n is $\frac{2}{3}\surd 2$ and the corresponding ray-pattern is shown in figure 1 (*a*). The lowest permissible value of Ω_n is $(\frac{2}{3})^{\frac{1}{2}}$ and the corresponding ray-pattern consists of the two parabolas

$$2z = (y \pm \Omega_1)^2. \tag{2.12}$$

An example of a closed ray-pattern at an intermediate value of ω is shown in figure 1 (*b*), for which $\omega = 0{\cdot}85$.

By referring either to figure 1 (*a*) or (*b*) it may be shown that any characteristic starting on the part OM of the line OI will reflect from the boundaries, changing sign and changing from one family to the other, but remaining within the domain $ABFEA$, and ultimately converges on the closed ray-pattern. As the rays converge on the limit cycle the motion of the fluid becomes more violent and exhibits a pathological behaviour in the limit. The pathology is worst as $\omega \to \omega_1^-$ and least as $\omega \to \Omega_1^+$, but on the other hand in the strict limit $\omega = \Omega_1$ the trapped mode is confined to just two lines and is physically unacceptable. In general, the fluid in the triangle GDC is undisturbed. Similar remarks can be made about the trapped modes in all the bands ($\Omega_n \leqslant \omega \leqslant \omega_n$) but for simplicity we shall restrict our attention to this particular one in this paper. Our object is to examine how a weak dissipation affects the structure of the solution, especially near the limit cycle, and since ray theory proved to be useful in the inviscid study we shall begin by examining how viscosity modifies disturbances confined to the immediate vicinity of a characteristic.

3. Fundamental solutions of the basic equation (2.9)

According to inviscid ray theory the disturbance is thought of as being confined to the immediate vicinity of a characteristic, for example the number $\xi_+ = c$ of the Γ_+ family, where

$$\xi_\pm = \tfrac{1}{2}(y \pm \omega)^2 - z, \tag{3.1}$$

and the property being carried on the ray is taken as

$$\int_{\xi_+ = c^-}^{\xi_+ = c^+} Z_+ \, dz = A_+, \tag{3.2}$$

A_+ being a constant and the same at all points of the ray between its intersections with the boundaries. Let us define this quantity to be the *flux* of the ray.

We can examine how these concepts are modified by the action of viscosity by looking at the fundamental solutions of the basic equations (2.9). In terms of y and $\xi_\pm$ these equations reduce to

$$\frac{1}{R}\frac{\partial^3 Z_\pm}{\partial \xi_\pm^3} = \mp 2i \frac{\partial Z_\pm}{\partial y}, \tag{3.3}$$

and acceptable fundamental solutions can be found on assuming that Z is of the form $|y|^{-\frac{1}{3}} Z(\eta)$, where $\eta = R^{\frac{1}{3}}(\xi_+ - c)/|y|^{\frac{1}{3}}$. For Z_+ we have

$$Z_{1+}(y, \xi_+) = -\frac{R^{\frac{1}{3}}}{\pi|y|^{\frac{1}{3}}}\int_0^\infty \exp\left[-i\eta t - \tfrac{1}{3}t^3\right] dt = -\frac{1}{\pi}\int_0^\infty \exp\left[-i(\xi_+ - c)\, t - \tfrac{1}{3}R^{-1}yt^3\right] dt \tag{3.4}$$

if $y > 0$ and $Z_{1+} = 0$ if $y < 0$. Another solution is

$$Z_{2+}(y, \xi_+) = \tilde{Z}_{1+}(-y, \xi_+), \tag{3.5}$$

where the tilde denotes the complex conjugate. For Z_- we have

$$Z_{1-}(y, \xi_-) = Z_{2+}(y, \xi_-), \quad Z_{2-} = Z_{1+}. \tag{3.6}$$

These solutions were first obtained by Morrison & Morgan (1956) and were discussed in detail by Moore & Saffman (1969), who showed that

$$\int_{-\infty}^{\infty} Z_{1+}(y, \xi_+)\, dz = 1 \quad \text{if} \quad y > 0 \tag{3.7}$$

and is zero if $y < 0$, with similar results for the other functions. Thus when $R \gg 1$ they can apparently be thought of as the generalization of the concentrated ray-pattern of inviscid theory to include the effect of a weak dissipation. However, they possess another important property which distinguishes them from the fundamental solutions normally encountered in fluid mechanics, in that they do not decay exponentially when $|\eta|$ is large. In fact

$$Z_{1+} = -\frac{R^{\frac{1}{3}}}{\pi|y|^{\frac{1}{3}}}\left[\frac{1}{i\eta} - \frac{3}{\eta^4} + O(\eta^{-7})\right] \quad \text{if} \quad y = 0. \tag{3.8}$$

Thus in the limit of zero viscosity

$$Z_{1+} = -\frac{1}{i\pi(\xi_+ - c)} \quad \text{if} \quad \xi_+ \neq c, \quad y > 0, \tag{3.9}$$

and so, in virtue of (3.7), may be regarded as having the form

$$-\left[\delta(\xi_+ - c) + 1/i\pi(\xi_+ - c)\right] \tag{3.10}$$

for $y > 0$, where δ is the Dirac delta function, with similar results for the other basic functions.

Consider now a concentrated disturbance in the neighbourhood of the Γ_+ characteristic $\xi_+ = c$, supposing that the fluid is inviscid. The appropriate solution is $Z_+ = f(\xi_+)$, $Z_- = 0$, where f vanishes except in the neighbourhood of $\xi_+ = c$ and is highly peaked at $\xi_+ = c$. No other restriction can be placed on f directly from the inviscid equations obtained by neglecting R^{-1} in (2.9) nor, at first sight, from the boundaries since they are effectively plane on the lateral scale of the disturbance. Now suppose that R^{-1} is small but not zero. Then the structure of the ray is dominated by the basic solutions Z_{1+} and Z_{2+} and we could write it as

$$Z_+ \sim A_1 Z_{1+}(y + y_1, \xi_+ - c) + A_2 Z_{2+}(y - y_2, \xi_+ - c), \tag{3.11}$$

where A_1, A_2, y_1 and y_2 are constants. Strictly, these source functions should be replaced by source integrals but since the ray is thin there is no advantage to be gained by greater precision. One relation between A_1 and A_2 may be found by integrating across the ray:

$$\int_{\xi_+=c^-}^{\xi_+=c^+} f\,dz = A_1+A_2, \tag{3.12}$$

provided that $y_2 > y > -y_1$, which is the case in the fluid if no sources are present there. The left-hand side is also constant from (3.2) and can be supposed given. The other relation between A_1 and A_2 comes from the condition that the ray is concentrated near $\xi_+ = c$ in the limit $R \to \infty$ and so the lateral tails implied by (3.9) must be absent. Hence, using (3.5)

$$A_1 - A_2 = 0, \tag{3.13}$$

and so the concentrated ray must be thought of as emanating from two sources with equal flux placed at points on the continuation of the ray beyond the fluid. If the boundary conditions are such that on any ray the property (3.13) does not hold then the ray theory described in S*a*, *b* and elsewhere (e.g. Keller & Mow 1969) may be inappropriate and it is possible that the lateral tails of the ray prevent the occurrence of trapped oscillations.

Equivalent results to these also follow if we think of the solution to the eigenvalue problem, defined by (2.9) with $R = \infty$ and with the associated boundary conditions, as a wave propagation, or initial-value, problem. Lighthill (1965, 1967) has shown that the physically relevant solution due to a periodic source of unit strength at $y = 0, \xi_+ = c$ is the one whose group velocity is directed away from the source and hence is given by (3.10) if $y > 0$ (Baines 1971), with corresponding results for Z_- and for $y < 0$. Again, therefore, we see that the disturbance consists of concentrated rays propagating along the characteristics through the source, together with four lateral tails whose strength depend only on distance from the characteristics and not on distance along the characteristics. Thus in order for the tails to be absent on any ray the sources must always occur in equal pairs just as in the viscous study.

4. Viscous reflexion of a concentrated ray

Let us now consider how a disturbance concentrated in the neighbourhood of a characteristic ray reflects from a boundary at the point P. It should be explained at the outset that the notion of reflexion is not strictly accurate. We have already seen that such a disturbance actually consists of two disturbances with equal fluxes travelling in opposite directions so that at the intersection only half the wave is incident, the other half having already been reflected. Thus the notion of reflexion is not entirely satisfactory and it is more appropriate instead to ask what disturbance concentrated near the other characteristic through P is compatible with the original disturbance and the conditions $u_\theta = u_\phi = u_r = 0$ on the boundary.

Suppose that the reflexion takes place in the neighbourhood of $y = \bar{y}$ in the plane $z = 0$. Then near this point the governing equations reduce to

$$R^{-1}\frac{\partial^3 Z_\pm}{\partial z^3} = 2ip_\pm\frac{\partial Z_\pm}{\partial z} \pm 2i\frac{\partial Z_\pm}{\partial y}, \tag{4.1}$$

where $p_\pm = \omega \pm \bar{y}$ and $|p_\pm| > 0$. We may take the two disturbances through the point $(\bar{y}, 0)$, but outside the boundary layer near $z = 0$, to be

$$\left.\begin{aligned} Z_+ &= \int_{-\infty}^{\infty} C_+(\alpha)\exp\{i\alpha(y - z/p_+)\}\,d\alpha, \\ Z_- &= \int_{-\infty}^{\infty} C_-(\alpha)\exp\{i\alpha(y + z/p_-)\}\,d\alpha, \end{aligned}\right\} \tag{4.2}$$

where, in view of the directions of propagation of the two half-waves $\alpha > 0$ and $\alpha < 0$ in each of Z_+ and Z_-, we regard C_+ as given if $\alpha < 0$ and C_- as given if $\alpha > 0$, and we want to determine the remaining properties of $C_\pm$. The rays are concentrated and so C_+ and C_- must both be continuous at $\alpha = 0$ and the scale for α must be large. On the other hand, the scale must not exceed $R^{\frac{1}{3}}$ for a larger scale implies a ray of thickness $\ll R^{-\frac{1}{3}}$ and this would immediately spread out under the action of viscosity.

In order to satisfy the boundary conditions at $z = 0$ we need to add to (4.2) solutions of (4.1) which decay exponentially as $z \to \infty$. Now

$$Z_\pm = \exp\{-\beta z + i\alpha y\} \quad \operatorname{Re}\beta > 0 \tag{4.3}$$

is a solution of (4.1) provided

$$\beta^3 - 2R(ip_\pm\beta \pm \alpha) = 0. \tag{4.4}$$

Of the three solutions of this cubic equation, one is unacceptable for all real α and one, β_2, is acceptable for all real α and corresponds to an Ekman layer. Thus

$$\beta_2 \approx (Rp_\pm)^{\frac{1}{2}}(1+i) \quad \text{when} \quad |\alpha| \ll (Rp_\pm)^{\frac{1}{2}}p_\pm$$

and either
$$\beta_2 \approx |2\alpha R|^{\frac{1}{3}} \quad \text{or} \quad |2\alpha R|^{\frac{1}{3}}e^{\frac{1}{3}\pi i} \tag{4.5}$$

when $|\alpha| \gg p_\pm(Rp_\pm)^{\frac{1}{2}}$ according as $\pm\alpha \gtrless 0$. The third solution β_3 is given by

$$\beta_3 \approx \pm\frac{i\alpha}{p_\pm} \mp \frac{\alpha^3}{2Rp_\pm^4} + \ldots \quad \text{when} \quad |\alpha| \ll (Rp_\pm)^{\frac{1}{2}}p_\pm;$$

when $|\alpha| \gg (Rp_\pm)^{\frac{1}{2}}p_\pm$

$$\beta_3 \approx \left\{\begin{aligned} &|2\alpha R|^{\frac{1}{3}}e^{\pm 2\pi i/3} \quad \text{if} \quad \pm\alpha > 0, \\ &|2\alpha R|^{\frac{1}{3}}e^{-\pi i/3} \quad \text{if} \quad \pm\alpha < 0. \end{aligned}\right\} \tag{4.6}$$

In virtue of the scaling requirements on α, the properties of β_2 and β_3 are only relevant when $|\alpha| < \sim R^{\frac{1}{3}}$. It appears at first sight that β_3 is unacceptable when $\mp\alpha > 0$ since then $\operatorname{Re}\beta_3 > 0$ but this is merely symptomatic of the fact that the corresponding solutions are advancing into the boundary. The solutions corresponding to these values of β_3 have already been provided by (4.2) and so, in order to complete the specification of $Z_\pm$ near $z = 0$, we only have to add the Ekman-layer solutions

$$Z_\pm = \int_{-\infty}^{\infty} D_\pm(\alpha)\exp\{i\alpha y - \beta_{2\pm}z\}\,d\alpha. \tag{4.7}$$

The argument now follows closely that given by Phillips (1963) earlier and we conclude that if the form of the ray on the Γ_+ characteristic and just outside the boundary layer is

$$Z_+ = F(y - z/p_+),$$

then the form of the ray on the Γ_- characteristic just outside the boundary layer is

$$Z_- = -F\left(y + \frac{z}{p_-}\right) + \frac{1+i}{2i}[(p_-^3 R)^{-\frac{1}{2}} + (p_+^3 R)^{-\frac{1}{2}}]F'\left(y - \frac{z}{p_-}\right). \tag{4.8}$$

The strengths of the two equal sources which dominate Z_+ may be computed using (3.7) and we see that, if

$$\int Z_+ \, dz = A_+ \tag{4.9}$$

taken across the ray on the Γ_+ characteristic, then

$$\int Z_- \, dz = A_- = A_+ p_-/p_+ \tag{4.10}$$

for the reflected ray, so that the sign of the flux is not changed but the magnitude is multiplied by the ratio p_-/p_+. This simple theory breaks down as $p_\pm \to 0$. A glance at (4.8) suggests that this occurs when $p_\pm^3 R \sim 1$, i.e. within a distance $R^{-\frac{1}{3}}$ of the critical circles $y = \pm\,\omega$, but this assumes that the function F, as distinct from its variable, is independent of $p_\pm$. Yang (1971) has also arrived at this conclusion; he also carried out experiments which confirm the reflexion law when $p_\pm \neq 0$. When $p_\pm = 0$ the above theory suggests that the flux of the reflected wave is zero; Yang has also carried out an experiment, in the analogous case of a stratified fluid, yielding agreement with this conclusion. In the present study, however, the representative width of the wave $= O(p_\pm)$ as $p_\pm \to 0$ since the flux is of this order and the maximum value of Z is conserved so long as viscous forces may be neglected. Hence this will no longer be justified when $(p_\pm^5 R)^{\frac{1}{2}} \sim 1$ since $F' \sim p_\pm^{-1} F$. Viscous forces must therefore be taken into account within a distance of $O(R^{-\frac{1}{5}})$ from the critical circles.

5. The neighbourhood of the limit cycle

It was shown in S*a*, *b* for trapped oscillations that the concentrated rays, as they reflect off the boundaries $z = 0, 1$ (not necessarily alternately, see figure 1), do not in general form a closed path but converge to a limit cycle. As they do so the bandwidth of the rays decreases until viscous forces can no longer be neglected in the ray. According to inviscid theory each of the functions $Z_\pm$ is quasi-periodic in the following way. Let the sth reflexion of a particular ray off the line $z = 0$ occur at $|y| = Y(\omega) - h_s$, where $Y(\omega)$ is a point on the limit cycle, so that $Y(\omega) = \omega_n$ if $\omega = \omega_n$. Further, let the bandwidth of the ray here be t_s. Then both t_s and h_s satisfy the law $t_{s+1} = \lambda(\omega)\, t_s$ where $\lambda < 1$, if $\Omega_n < \omega < \omega_n$ and $t_{s+1} = \mu_n t_s^2$ if $\omega = \omega_n$, where μ_n is a constant. Thus the bandwidth decreases as the limit cycle is approached but, so long as viscous forces may be neglected, the maximum value of $|\psi|$ is constant. This process changes when $t_s \sim R^{-\frac{1}{3}}$ as then viscous forces come into play to prevent any further reduction in bandwidth. At this stage the velocity of the fluid is of $O(Vt_s^{-1}) \sim O(VR^{\frac{1}{3}})$, where V is a representative velocity in the trapped oscillation away from the limit cycle; we can expect that this is a maximum and that the velocity then decays to zero within a distance of $O(R^{-\frac{1}{3}})$ of the limit cycle, but on the other side. The action of viscosity near the limit cycle causes a loss of energy by dissipation. Referring back to (2.1) we can easily see that if T is the total kinetic energy of the relative motion

$$\frac{dT}{dt} = -\nu \iint (\mathbf{u}\,.\operatorname{curl}\operatorname{curl}\mathbf{u})\,dS = -\nu \iint (\operatorname{curl}\mathbf{u})^2\,dS \tag{5.1}$$

taken over the disturbed region. The right-hand side is significant only near the limit cycle and so we have

$$\frac{1}{\Omega\epsilon^{\frac{1}{2}}T}\frac{dT}{dt} \sim R^{-1}R^{\frac{2}{3}}R^{\frac{2}{3}}R^{-\frac{1}{3}} \sim 1; \tag{5.2}$$

according to this argument the decay is therefore on an inviscid scale. It should be borne in mind, however, that the argument is coarse and it may well be that the decay rate does tend to zero as $R \to \infty$, but more slowly than any negative power of R. An argument in support of this view is given in the next section.

The structure of the disturbance near the limit cycle also enables us to sharpen the condition on Δ for the validity of the linearized theory. The relevant requirement is that all velocity gradients should be small and hence, using the above estimates,

$$\Delta R^{\frac{2}{3}} \ll \epsilon, \tag{5.3}$$

which is severe.

In order to explore further the nature of the disturbance when viscous effects are important we must distinguish between oscillations with $\omega \neq \omega_n$ and those with $\omega = \omega_n$. If $\omega \neq \omega_n$, as figure 1 (*b*) illustrates, all the characteristics intersect the boundaries at a finite angle and so the theory developed in §§ 3 and 4 is sufficient for the discussion of the history of the disturbance near the limit cycle. Once the bandwidth of the disturbance is reduced to $\sim R^{-\frac{1}{3}}$ the action of viscosity prevents it from getting any thinner but at each reflexion it moves nearer to the limit cycle and so there must be some spillage. One can expect in fact that, if the disturbance coming down the Γ_+ characteristics to $z = 0$ is confined to the region $\omega - R^{-\frac{1}{3}} < y < \omega$ just outside the Ekman layer at $z = 0$, the reflected ray will have the same flux $(\sim R^{-\frac{1}{3}})$ and spread out under the action of viscosity so that on returning to the neighbourhood a substantial part of the flux will be outside the limit cycle and tend to be lost to the trapped region in subsequent reflexions. It seems therefore that there will be a loss of flux due to leakage $\sim R^{-\frac{1}{3}}$. The energy associated with this leakage is however of $O(R^{\frac{1}{3}})$, which partly accounts for the rapid decay of the free oscillations.

An important assumption underlying our discussions here is that the disturbance can be thought of as concentrated along rays and for this to be possible they must be regarded as two disturbances of equal flux but travelling in opposite directions. While this notion seems to lead to consistent conclusions over the majority of the trapped zone, we note two difficulties near the limit cycle. First, the wave escaping from the trapped region is uncompensated, unless we permit an inflow of energy, and hence has a lateral tail which, to begin with, is significant inside the trapped region. Secondly, the decay of the disturbance discussed here must be associated with a birth of the corresponding ray travelling in the opposite direction. Unless there is a source of flux on the limit cycle, therefore, the oscillation cannot be maintained. Nor can it apparently be confined within the limit cycle, even approximately, when $R \gg 1$ for there is a continual leakage of flux from the trapped zone and a dissipation of energy, roughly of the order of that available, in the neighbourhood of the limit cycle.

This argument no longer applies when $\omega = \omega_n$, so that the limit cycle touches the inner sphere since $p_\pm$ is small near the limit cycle and consequently the theory of § 4 needs modification. The formal solution of the reflexion problem is difficult in these cases and has not yet been achieved. However, the qualitative features can be seen directly and in view of the negative conclusions these are probably sufficient for our purposes. When the concentrated disturbance is within a distance of $O(R^{-\frac{1}{6}})$ of the limit cycle the reflexion at $(\pm\omega_n, 0)$ is seriously modified by viscosity and a significant part of the flux is lost to the trapped region by being carried out on the characteristic Γ_0 which touches the plane $z = 0$, i.e.

$$(y \pm \omega_n)^2 - 2z = 0,\ y > \omega_n.$$

The remainder is concentrated into a ray of width $\sim R^{-\frac{2}{5}}$ near the limit cycle; it is therefore modified by viscosity such that it increases its thickness to $R^{-\frac{1}{3}}$ and on return to the neighbourhood of $z = 0$ loses more flux on the Γ_0 characteristic. Thus the main difference between this case and the others is that here the flux lost is immediately carried right away from the trapped zone whereas in the others the rays carrying the disturbance out move only gradually away from the limit cycle. In all cases the rays have a lateral tail unless compensated for.

6. A forced oscillation

The arguments of the previous section strongly suggest that, even in the limit $R \to \infty$, it is unlikely that free oscillations, confined within a finite distance of the line $y = 0$, can occur in the form usually considered. If set up they would decay at an inviscid rate near the limit cycle, so that their energy would be drained off, and in addition there would be a substantial leakage of flux across the cycle owing to the action of viscosity even though $R \gg 1$. Finally the leakage of flux would have a lateral tail since there is no compensating inflow of flux to the trapped region.

One possible way in which trapped oscillations might be set up is by imposing a forced oscillation on the boundaries, for example if they were given a normal velocity

$$\sim \Delta\epsilon^{\frac{1}{2}} \exp\{2i\Omega\epsilon^{\frac{1}{2}}\omega t\}. \tag{6.1}$$

Such a notion might lead to difficulties with the equation of continuity but these can be avoided if the fluid is regarded as being slightly compressible. Alternatively we may think of the normal velocity as equivalent to the Ekman pumping due to a differential rotation, which from (5.3) must be $\ll \epsilon^{\frac{1}{4}}R^{-\frac{1}{6}}\Omega$.

If we now apply the inviscid theory of wave motion to this problem we see that the boundary conditions of § 2, namely $f + g = 0$ on $z = 0, 1$, must be changed to $f+g \propto y$ on $z = 0, 1$. Suppose now we follow a particular ray as it reflects off each boundary in turn, the values of f or g changing by a finite amount at each reflexion. If no lateral tails are present in any neighbourhood none will be developed by the reflexion except possibly, and exceptionally, when a characteristic touches the inner sphere. In general therefore the use of ray theory seems consistent and we can expect the magnitudes of f and g to remain finite at finite values of y. We would not expect there to be any resonance effect when R is finite and is allowed to increase indefinitely.

The situation is different when ω lies within one of the ranges $\Omega_n < \omega < \omega_n$ of continuous spectra for trapped oscillations in the inviscid theory, for now the rays can reflect off the boundaries an infinite number of times, while remaining within a finite distance of the origin, as they approach the limit cycle. In the viscous theory, provided that the tails cancel, the number of reflexions is limited by the onset of viscous effects when the ray is $\sim R^{-\frac{1}{3}}$ from the cycle. An estimate of the maximum number of reflexions can be made by noting that if the successive reflexions of the ray from $z = 0$ occur at distances h_s from the limit cycle, $h_{s+1} \approx \lambda(\omega)\, h_s$ when $h_s \ll 1$, where $\lambda(\Omega_n) = 1$ and $\lambda(\omega_n) = 0$. We can expect therefore that it will take

$$\log R/3 \log \lambda \tag{6.2}$$

reflexions from $z = 0$ to reach the limit cycle and pass over to the other side and away from the trapped zone. It follows that the disturbances in the trapped region will be larger than those outside by a factor $\log R/\log \lambda$.

When $\omega = \omega_n$, the limit cycle touches the line $z = 0$, $\lambda = 0$, $h_{s+1} \propto h_s^2$ and the appropriate factor is $\sim \log\log R$. Once through the limit cycle there is then a more rapid dispersion of the disturbance owing to the reflexion at $z = 0$. When $\omega = \Omega_n$, the limit cycle encloses a region of zero measure and for neighbouring rays $h_{s+1} = h_s + O(h_s^2)$. Hence $h_s \propto s^{-1}$ and the appropriate magnification of the disturbance $\sim R^{\frac{1}{3}}$. For values of ω just smaller than Ω_n we can estimate the maximum order of magnitude of the disturbance by considering how many reflexions are then needed to cross the characteristic through $(\Omega_n, 1)$. If h_s is defined as before it is easy to show that

$$h_{s+1} = h_s - A_1 h_s^2 - A_2(\Omega_n - \omega),$$

where A_1 and A_2 are positive constants. It then follows that $(\Omega_n - \omega)^{-\frac{1}{2}}$ reflexions are required to pass through the characteristic and this suggests that the maximum amplitude of the disturbance is likely to be of the same order.

In view of the weakness of the dependence of the maximum disturbance on R when $\omega \neq \Omega_n$ and of the bizarre nature of the oscillations when $\omega = \Omega_n$ it is unlikely that recognizable trapped oscillations would appear in a practical situation even if they were valid limits as $R \to \infty$. The most we would infer is that when $\Omega_n < \omega < \omega_n$ some amplification will take place as the rays carrying the disturbance cross the appropriate limit cycle. There is also a slight preference for a recognizable trapped mode being obtained if ω is near Ω_n.

The only numerical studies of trapped oscillations in a spherical shell known to the author are due to Israeli (1971), who considered the motion engendered when the outer sphere is given a small sinusoidal angular velocity in addition to the basic rigid rotation. When $R \gg 1$ one might expect that the motion would be inviscid except in the Ekman boundary layers and near the limit cycle. His results indicate that there is a weak resonance at values of ω in the frequency range $\Omega_1 \leqslant \omega \leqslant \omega_1$ and while at $E^{-1} = 16000$ the maximum occurs at $\omega \approx \Omega_1$ in agreement with the above argument, at $E^{-1} = 36000$ it occurs at $\omega \approx \omega_1$. The graphs of the stream function ψ displayed clearly indicate the presence of an Ekman boundary layer near $z = 1$, but no sign of the limit cycle of the inviscid theory, although there are some indications of a trapped oscillation. One is inclined to suspect that the difficulties inherent in the inviscid theory are the reason but it should be noted that even when $E^{-1} = 36000$ and $\epsilon = \frac{1}{7}$, which is the nearest Israeli was able to get to the conditions required, $R \approx 200$. For the shell with $a = 2b$ and $\epsilon = 1$, Israeli obtained a sharp resonance when $E^{-1} = 10^4$, which agreed well with Aldridge's experiments.

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